Classification of integrability and non-integrability for some quantum spin chains

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Contents

Definitions

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by the number of local conserved quantities

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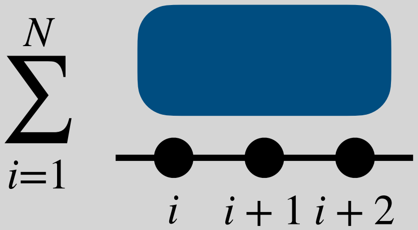
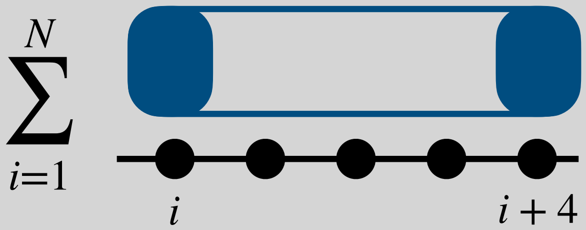
(spatially) ***k*-local quantity**:
A sum of operators acting on l consecutive sites with $l \leq k$

ex.) $\sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^y \sigma_{i+2}^z$: 3-local



$\sum_{i=1}^{N} \sigma_i^z \sigma_{i+4}^x$: 5-local



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A sum of operators acting on l consecutive sites with $l \leq k$

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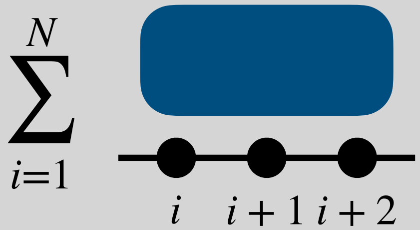

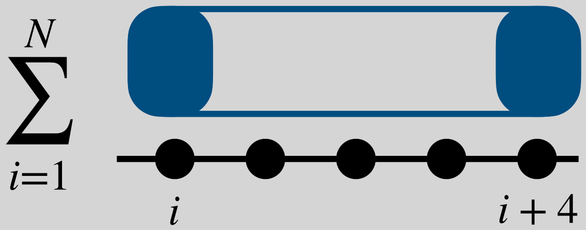

Local conserved quantity:
An $O(1)$-local quantity which commutes with H

ex.) $Q = \sum_{i=1}^{N} \vec{\sigma}_i \cdot (\vec{\sigma}_{i+1} \times \vec{\sigma}_{i+2})$ for $H = \sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$

Definitions

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# of local conserved quantities

∞ **Integrable systems**

0 **Non-integrable systems**

(excluding trivial ones)

Definitions

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# of local conserved quantities

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finite Partially integrable systems

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(excluding trivial ones)

Most systems are non-integrable

It is strongly expected that **non-integrability is ubiquitous**

- Consistency with empirical laws of macro. systems (linear response theory, thermalization, heat conduction, etc.)
- Numerical simulations (energy spectra)

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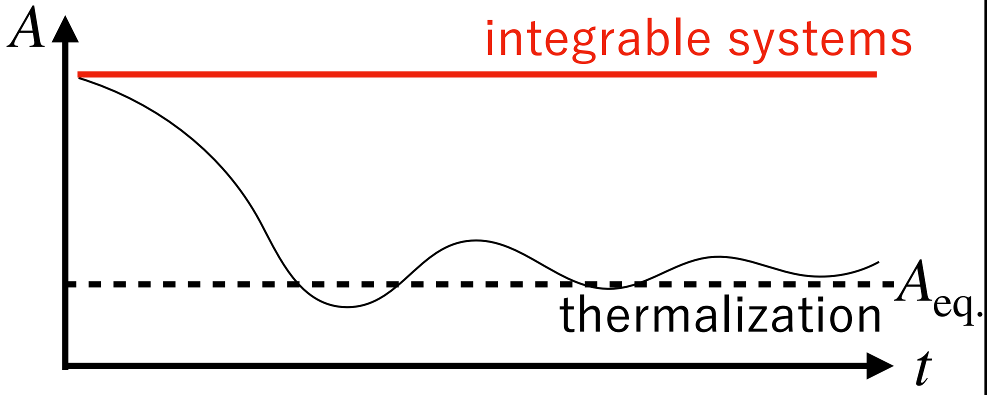

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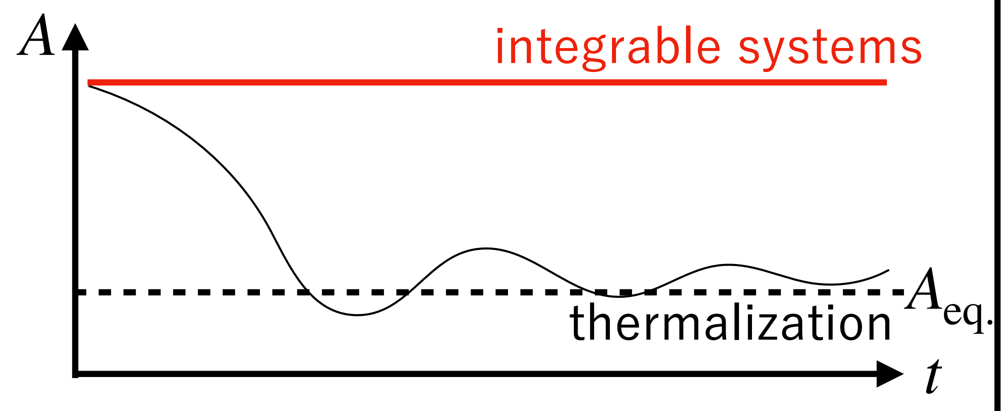

But rigorous treatment of non-integrability is difficult…
(It was out of scope of mathematical physics until 2019)

Proof of non-integrability

Recently, some models were **shown to be non-integrable**

XYZh model [Shiraishi (2019)]

$$H = \sum_{i=1}^{N} (J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1} + h Z_i)$$

is non-integrable if $J_X, J_Y, J_Z \neq 0, J_X \neq J_Y, h \neq 0$

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Mixed-field Ising model [Chiba (2024)]

$$H = \sum_{i=1}^{N} (Z_i Z_{i+1} + h_X X_i + h Z_i)$$

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PXP model [Park, Lee (arXiv:2403)]

n.n.n. Heisenberg model [Shiraishi (2024)]

Research question

Non-integrability proof studies have begun,
but these studies have dealt with **individual** systems

Classification of integrability and non-integrability
for **general classes** is still lacking

It is not proved that **non-integrability is ubiquitous**

Contents

Model

General spin-1/2 chains with symmetric n.n. interaction

$$H = \sum_{i=1}^{N} \left(\sum_{\alpha,\beta} J_{\alpha\beta} \sigma_i^\alpha \sigma_{i+1}^\beta + \sum_\alpha h_\alpha \sigma_i^\alpha \right)$$

with $J_{\alpha\beta} = J_{\beta\alpha}$

Model

General spin-1/2 chains with symmetric n.n. interaction

$$H = \sum_{i=1}^{N} \begin{pmatrix} J_{XX}\, X_i X_{i+1} + J_{XY}\, X_i Y_{i+1} + J_{XZ}\, X_i Z_{i+1} \\ + J_{YX}\, Y_i X_{i+1} + J_{YY}\, Y_i Y_{i+1} + J_{YZ}\, Y_i Z_{i+1} \\ + J_{ZX}\, Z_i X_{i+1} + J_{ZY}\, Z_i Y_{i+1} + J_{ZZ}\, Z_i Z_{i+1} \\ h_X\, X_i + h_Y\, Y_i + h_Z\, Z_i \end{pmatrix}$$

(Abbreviation: $\sigma_i^x \to X_i$ etc.)

with $J_{\alpha\beta} = J_{\beta\alpha}$

Model

General spin-1/2 chains with symmetric n.n. interaction

$$H = \sum_{i=1}^{N} \begin{pmatrix} J_{XX} X_i X_{i+1} + J_{XY} X_i Y_{i+1} + J_{XZ} X_i Z_{i+1} \\ + J_{YX} Y_i X_{i+1} + J_{YY} Y_i Y_{i+1} + J_{YZ} Y_i Z_{i+1} \\ + J_{ZX} Z_i X_{i+1} + J_{ZY} Z_i Y_{i+1} + J_{ZZ} Z_i Z_{i+1} \\ h_X X_i + h_Y Y_i + h_Z Z_i \end{pmatrix}$$

(Abbreviation: $\sigma_i^x \to X_i$ etc.) with $J_{\alpha\beta} = J_{\beta\alpha}$

Including:

-Integrable systems: Heisenberg, Transverse-field Ising, etc.

-Non-integrable systems: XYZh, Mixed-field Ising

Result

Main Theorem [Yamaguchi, Chiba, Shiraishi arXiv:2411.02162]
All models in this class are **non-integrable** (**do not have** nontrivial **local conserved quantities**), (nontrivial: $k \geq 3$)
except for known integrable systems and their equivalents

Result

Main Theorem [Yamaguchi, Chiba, Shiraishi arXiv:2411.02162]
All models in this class are **non-integrable** (**do not have** nontrivial **local conserved quantities**), (nontrivial: $k \geq 3$)
except for known integrable systems and their equivalents

Implications

- **Ubiquitousness of non-integrability**
- No overlooked integrable systems
- No partially integrable systems

Contents

Global spin rotation

$$H = \sum_{i=1}^{N} \left(\begin{pmatrix} J_{XX} X_i X_{i+1} + J_{XY} X_i Y_{i+1} + J_{XZ} X_i Z_{i+1} \\ + J_{YX} Y_i X_{i+1} + J_{YY} Y_i Y_{i+1} + J_{YZ} Y_i Z_{i+1} \\ + J_{ZX} Z_i X_{i+1} + J_{ZY} Z_i Y_{i+1} + J_{ZZ} Z_i Z_{i+1} \end{pmatrix} + (h_X X_i + h_Y Y_i + h_Z Z_i) \right)$$

with $J_{\alpha\beta} = J_{\beta\alpha}$

Global spin rotation

$$J = \begin{pmatrix} J_{XX} & J_{XY} & J_{XZ} \\ J_{YX} & J_{YY} & J_{YZ} \\ J_{ZX} & J_{ZY} & J_{ZZ} \end{pmatrix}$$ **can be diagonalized**

by global spin rotation $\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = R \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$, because J is real symmetric

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keeping (non-)integrability

$$H = \sum_{i=1}^{N} (J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i)$$

Division into cases

We treat H separately, depending on **<u>rank of J</u>**

(the number of nonzero elements in $\{J_X, J_Y, J_Z\}$)

rank 0: $$H = \sum_{i=1}^{N} h_Z Z_i$$

with $J_X, J_Y, J_Z \neq 0$

rank 1: $$H = \sum_{i=1}^{N} \left(J_Z Z_i Z_{i+1} + h_X X_i + h_Z Z_i\right)$$

rank 2: $$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right)$$

rank 3: $$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right)$$

Classification table

Integrable or Non-integrable

rank	H	detail conditions	
0	$\sum_{i=1}^{N} h_Z Z_i$		I (trivial)
1	$\sum_{i=1}^{N} \begin{pmatrix} J_Z Z_i Z_{i+1} \\ +h_X X_i \\ +h_Z Z_i \end{pmatrix}$	$h_X = 0$	I (trivial)
		$h_X \neq 0, h_Z = 0$	I transverse field Ising
		$h_X \neq 0, h_Z \neq 0$	N [Chiba]
2	$\sum_{i=1}^{N} \begin{pmatrix} J_X X_i X_{i+1} \\ +J_Y Y_i Y_{i+1} \\ +h_X X_i \\ +h_Z Z_i \\ +h_Z Z_i \end{pmatrix}$	$(h_X, h_Y) = (0,0)$	I XY
		$(h_X, h_Y) \neq (0,0)$	N our result

Classification table

Integrable (I) or Non-integrable (N)

rank	H	detail conditions			
3	$\sum_{i=1}^{N}\begin{pmatrix} J_X X_i X_{i+1} \\ +J_Y Y_i Y_{i+1} \\ +J_Z Z_i Z_{i+1} \\ +h_X X_i \\ +h_Z Z_i \\ +h_Z Z_i \end{pmatrix}$	$J_X = J_Y = J_Z$		I	XXX (Heisenberg)
		$J_X = J_Y \neq J_Z$ (two are equal)	$(h_X, h_Y) = (0,0)$	I	XXZ
			$(h_X, h_Y) \neq (0,0)$	N	**our result**
		J_X, J_Y, J_Z all different	$(h_X, h_Y, h_Z) = \vec{0}$	I	XYZ
			$(h_X, h_Y, h_Z) \neq \vec{0}$	N	**our result**

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Idea of non-integrability proof

[Shiraishi (2019)]

Expansion of general local quantity

$$Q = \sum_{\mathbb{A}} q_{\mathbb{A}} \mathbb{A}$$

($\{\mathbb{A}\}$: basis of local quantities)

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= Absence of (nontrivial) local conserved quantity Q
= **Absence of** (nontrivial) **solution to the equations of** $\{q_{\mathbb{A}}\}$

Basis of local quantities

Basis of local quantities $\{\mathbb{A}\}$ = **local Pauli strings**

Pauli string: tensor product of Pauli operator of each site

$$\bigotimes_{i=1}^{N} \{X_i, Y_i, Z_i, I_i\}$$

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ex.) Basis of 1-local quantities

$I,$ $\quad X_i, Y_i, Z_i : \boldsymbol{A}_i^1$ $\quad$ for each i

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ex.) Basis of 2-local quantities

$$I, \qquad X_i, Y_i, Z_i : \boldsymbol{A}_i^1 \qquad \text{for each } i$$

$$X_i X_{i+1}, X_i Y_{i+1}, \ldots, Z_i Z_{i+1} : \boldsymbol{A}_i^2$$

Basis of local quantities

Basis of local quantities $\{\mathbb{A}\}$ = **local Pauli strings**

Pauli string: tensor product of Pauli operator of each site

$$\bigotimes_{i=1}^{N} \{X_i, Y_i, Z_i, I_i\}$$

ex.) Basis of 3-local quantities

$$I, \qquad X_i, Y_i, Z_i : \boldsymbol{A}_i^1 \qquad \text{for each } i$$

$$X_iX_{i+1}, X_iY_{i+1}, \ldots, Z_iZ_{i+1} : \boldsymbol{A}_i^2$$

$$X_iX_{i+1}X_{i+2}, X_iX_{i+1}Y_{i+2}, \ldots, Z_iI_{i+1}Z_{i+2} : \boldsymbol{A}_i^3$$

Commutator of basis elements

How is the linear map $[\,\bullet\,, H]$ represented?

ex.) $[X_i Y_{i+1} Z_{i+2}\,, Y_{i-1} Y_i]$

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ex.) $[X_i Y_{i+1} Z_{i+2}, Y_{i-1} Y_i]$

$$= Y_{i-1} [X_i, Y_i] Y_i Z_{i+2}$$

$$= +2i\, Y_{i-1} Z_i Y_i Z_{i+2}$$

Commutators of Pauli strings are other Pauli strings (or 0)

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Column expression of this commutator:

		X_i	Y_{i+1}	Z_{i+2}
	Y_{i-1}	Y_i		
$+2i$	Y_{i-1}	Z_i	Y_{i+1}	Z_{i+2}

Commutator of basis elements

How is the linear map $[\bullet, H]$ represented?

ex.) $[\underbrace{X_i Y_{i+1} Z_{i+2}}_{A_i^3 \text{ in } Q}, \underbrace{Y_{i-1} Y_i}_{\text{in } H}]$

$= Y_{i-1} [X_i, Y_i] Y_i Z_{i+2}$

$= +2i\, Y_{i-1} Z_i Y_i Z_{i+2}$ ← B_{i-1}^4 in $[Q, H]$

Commutators of Pauli strings are other Pauli strings (or 0)

Column expression of this commutator:

		X_i	Y_{i+1}	Z_{i+2}		$: A_i^3$
	Y_{i-1}	Y_i				
$+2i$	Y_{i-1}	Z_i	Y_{i+1}	Z_{i+2}		$: B_{i-1}^4$

Idea of non-integrability proof

Expansion of general k-local quantity

$[\bullet, H]$: linear map

$$Q = \sum_{l=1}^{k} \sum_{\boldsymbol{A}_i^l} q_{\boldsymbol{A}_i^l} \boldsymbol{A}_i^l$$

($\{\boldsymbol{A}_i^l\}$: basis of local quantities)
$\{\boldsymbol{B}_i^l\}$

$$[Q, H] = \sum_{l=1}^{k+1} \sum_{\boldsymbol{B}_i^l} r_{\boldsymbol{B}_i^l} \boldsymbol{B}_i^l$$

$r_{\boldsymbol{B}_i^l}$: linear combination of $\{q_{\boldsymbol{A}_i^l}\}$

Conservation condition: $r_{\boldsymbol{B}_i^l} = 0$ for all $\boldsymbol{B}_i^l$

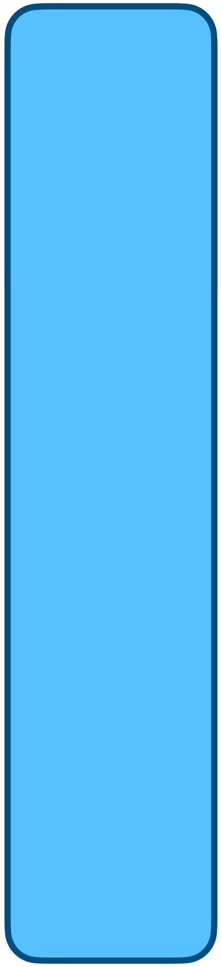
A system of linear equations of $\{q_{\boldsymbol{A}_i^l}\}$

Non-integrability
= Absence of (nontrivial) local conserved quantity Q
= **Absence of** (nontrivial) **solution to the equations of** $\{q_{\boldsymbol{A}_i^l}\}$

Idea of non-integrability proof

We will **prove** $q_{\boldsymbol{A}_i^k} = 0$ **for all** $\boldsymbol{A}_i^k$ for k-local conserved quantity Q

$$
\begin{aligned}
Q = & \sum_{\boldsymbol{A}_i^1} q_{\boldsymbol{A}_i^1} \boldsymbol{A}_i^1 \\
& + \sum_{\boldsymbol{A}_i^2} q_{\boldsymbol{A}_i^2} \boldsymbol{A}_i^2 \\
& + \sum_{\boldsymbol{A}_i^3} q_{\boldsymbol{A}_i^3} \boldsymbol{A}_i^3 \\
& + \cdots \\
& + \sum_{\boldsymbol{A}_i^{k-1}} q_{\boldsymbol{A}_i^{k-1}} \boldsymbol{A}_i^{k-1} \\
& + \sum_{\boldsymbol{A}_i^k} {\color{red} q_{\boldsymbol{A}_i^k}} \boldsymbol{A}_i^k
\end{aligned}
$$



k-local
conserved quantity

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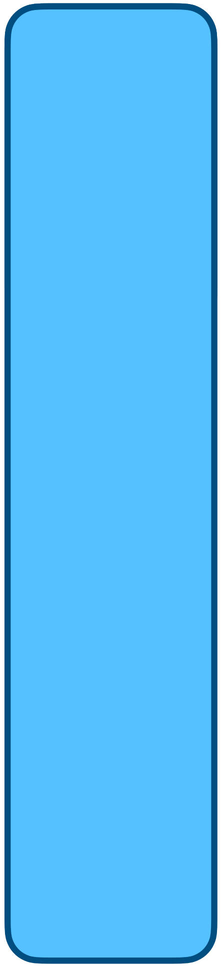
We will **prove** $q_{\boldsymbol{A}_i^k} = 0$ **for all** $\boldsymbol{A}_i^k$ for k-local conserved quantity Q

$$
\begin{aligned}
Q = &\sum_{\boldsymbol{A}_i^1} q_{\boldsymbol{A}_i^1} \boldsymbol{A}_i^1 \\
&+ \sum_{\boldsymbol{A}_i^2} q_{\boldsymbol{A}_i^2} \boldsymbol{A}_i^2 \\
&+ \sum_{\boldsymbol{A}_i^3} q_{\boldsymbol{A}_i^3} \boldsymbol{A}_i^3 \\
&+ \cdots \\
&+ \sum_{\boldsymbol{A}_i^{k-1}} q_{\boldsymbol{A}_i^{k-1}} \boldsymbol{A}_i^{k-1} \\
&+ \sum_{\boldsymbol{A}_i^k} 0 \, \boldsymbol{A}_i^k
\end{aligned}
$$



k-local conserved quantity

Idea of non-integrability proof

We will **prove** $q_{A_i^k} = 0$ **for all** A_i^k for k-local conserved quantity Q

$$
\begin{aligned}
Q = &\sum_{A_i^1} q_{A_i^1} A_i^1 \\
&+ \sum_{A_i^2} q_{A_i^2} A_i^2 \\
&+ \sum_{A_i^3} q_{A_i^3} A_i^3 \\
&+ \cdots \\
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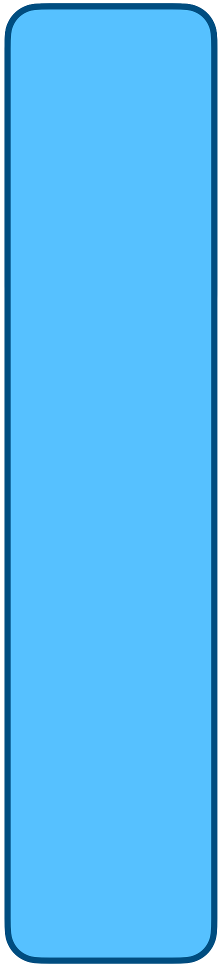
$$



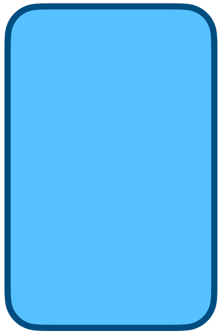
k-local conserved quantity

absent (for general $3 \le k \le N/2$)

Idea of non-integrability proof

We will **prove** $q_{\boldsymbol{A}_i^k} = 0$ **for all** $\boldsymbol{A}_i^k$ for k-local conserved quantity Q

$$Q = \sum_{\boldsymbol{A}_i^1} q_{\boldsymbol{A}_i^1} \boldsymbol{A}_i^1 + \sum_{\boldsymbol{A}_i^2} q_{\boldsymbol{A}_i^2} \boldsymbol{A}_i^2$$



trivial local conserved quantity ($k \leq 2$, such as H)

k-local conserved quantity

absent (for general $3 \leq k \leq N/2$)

Contents

Proof

2	$\sum_{i=1}^{N}\begin{pmatrix} J_X X_i X_{i+1} \\ +J_Y Y_i Y_{i+1} \\ +h_X X_i \\ +h_Z Z_i \\ +h_Z Z_i \end{pmatrix}$	$(h_X, h_Y) = (0{,}0)$	I	XY
		$(h_X, h_Y) \neq (0{,}0)$	N	**our result**

$$J_X \neq 0\,, J_Y \neq 0$$

Proof

2	$\sum_{i=1}^{N}\begin{pmatrix} J_X X_i X_{i+1} \\ +J_Y Y_i Y_{i+1} \\ +h_X X_i \\ +h_Z Z_i \\ +h_Z Z_i \end{pmatrix}$	$(h_X, h_Y) = (0,0)$	I	XY
		$(h_X, h_Y) \neq (0,0)$	N	our result

$J_X \neq 0\,, J_Y \neq 0$

Today I present the proof of rank 2

Proof: $A^k = Z \cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

We can immediately show that $\boldsymbol{A}^k = Z \overbrace{\cdots}^{\text{any}} X$ has zero coeff.

$\boldsymbol{A}_i^k$: $Z_i \quad \cdots \quad X_{i+k-1}$

Proof: $A^k = Z \cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

We can immediately show that $\boldsymbol{A}^k = Z \overbrace{\cdots}^{\text{any}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z \cdots ZY$,

which is generated only by the following commutator:

$$\begin{array}{lccccc} \boldsymbol{A}_i^k : & & Z_i & \cdots & X_{i+k-1} & \\ & & & & Y_{i+k-1} & Y_{i+k} \\ \hline \boldsymbol{B}_i^{k+1} : & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

Proof: $A^k = Z\cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

We can immediately show that $\boldsymbol{A}^k = Z \overbrace{\cdots}^{\text{any}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZY$,

which is generated only by the following commutator:

$$\begin{array}{lccccc} \boldsymbol{A}_i^k : & & Z_i & \cdots & X_{i+k-1} & \\ & & & & Y_{i+k-1} & Y_{i+k} \\ \hline \boldsymbol{B}_i^{k+1} : & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

$$\begin{array}{ccccc} & ? & \cdots & ? & \\ ? & ? & & & \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} & \end{array} \qquad \begin{array}{cccc} ? & \cdots & ? & \\ & & ? & ? \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

Proof: $A^k = Z\cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

We can immediately show that $\boldsymbol{A}^k = Z \overbrace{\cdots}^{\text{any}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZY$,

which is generated only by the following commutator:

$$\begin{array}{lccccc} \boldsymbol{A}_i^k : & & Z_i & \cdots & X_{i+k-1} & \\ & & & & Y_{i+k-1} & Y_{i+k} \\ \hline \boldsymbol{B}_i^{k+1} : & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

$$\begin{array}{ccccc} & ? & \cdots & ? & \\ Z & ? & & & \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} & \end{array} \qquad \begin{array}{cccc} ? & \cdots & ? & \\ & & ? & ? \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

Proof: $A^k = Z\cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0{,}0) \end{matrix}$$

We can immediately show that $\boldsymbol{A}^k = Z \overbrace{\cdots}^{\text{any}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZY$,

which is generated only by the following commutator:

$$\begin{array}{lcccccc} \boldsymbol{A}_i^k : & & Z_i & \cdots & X_{i+k-1} & \\ & & & & Y_{i+k-1} & Y_{i+k} \\ \hline \boldsymbol{B}_i^{k+1} : & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

$$\begin{array}{ccccc} & ? & \cdots & ? & \\ Z & ? & & & \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} & \end{array} \qquad \begin{array}{cccc} ? & \cdots & ? & \\ & & ? & Y \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

Proof: $A^k = Z \cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

We can immediately show that $\boldsymbol{A}^k = Z \overbrace{\cdots}^{\text{any}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z \cdots ZY$,

which is generated only by the following commutator:

$$\begin{array}{lccccc}
\boldsymbol{A}_i^k : & & Z_i & \cdots & X_{i+k-1} & \\
 & & & & Y_{i+k-1} & Y_{i+k} \\
\hline
\boldsymbol{B}_i^{k+1} : & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k}
\end{array}$$

$$\begin{array}{ccccc}
 & ? & \cdots & ? & \\
Z & ? & & & \\
\hline
Z_i & \cdots & Z_{i+k-1} & Y_{i+k} &
\end{array}
\qquad
\begin{array}{cccc}
? & \cdots & ? & \\
 & & Y & Y \\
\hline
Z_i & \cdots & Z_{i+k-1} & Y_{i+k}
\end{array}$$

Proof: $A^k = Z \cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

We can immediately show that $\boldsymbol{A}^k = Z \overset{\text{any}}{\overbrace{\cdots}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z \cdots Z Y$,

which is generated only by the following commutator:

$$\begin{array}{lccccc} \boldsymbol{A}_i^k : & & Z_i & \cdots & X_{i+k-1} & \\ & & & & Y_{i+k-1} & Y_{i+k} \\ \hline \boldsymbol{B}_i^{k+1} : & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

✗

$$\begin{array}{ccccc} & ? & \cdots & ? & \\ Z & ? & & & \\ \hline & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

✗

$$\begin{array}{cccc} Z & \cdots & X & \\ & & Y & Y \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

Proof: $A^k = Z\cdots X$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

We can immediately show that $\boldsymbol{A}^k = Z \overset{\text{any}}{\overbrace{\cdots}} X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZY$,

which is generated only by the following commutator:

$$\begin{array}{lccccc} \boldsymbol{A}_i^k: & & Z_i & \cdots & X_{i+k-1} & \\ & & & & Y_{i+k-1} & Y_{i+k} \\ \hline \boldsymbol{B}_i^{k+1}: & +2i & Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

$$+2i\, q_{(Z\cdots X)_i} J_Y = 0$$

$$\rightarrow \quad q_{(Z\cdots X)_i} = 0$$

❌

$$\begin{array}{cccc} & ? & \cdots & ? \\ Z & ? & & \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

❌

$$\begin{array}{cccc} Z & \cdots & X & \\ & & Y & Y \\ \hline Z_i & \cdots & Z_{i+k-1} & Y_{i+k} \end{array}$$

Proof: $A^k = Z\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \text{ with } \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

By similar discussions,

$\boldsymbol{A}^k = Z\cdots Y$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZX$

$$\begin{array}{ccccc} & Z & \cdots & Y & \\ & & & X & X \\ \hline -2i & Z & \cdots & Z & X \end{array}$$

$\boldsymbol{A}^k = Z\cdots Z$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots YX$

$$\begin{array}{ccccc} & Z & \cdots & Z & \\ & & & X & X \\ \hline +2i & Z & \cdots & Y & X \end{array}$$

Proof: $A^k = Z\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

By similar discussions,

$\boldsymbol{A}^k = Z\cdots Y$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZX$

$$\begin{array}{rcccc} & Z & \cdots & Y & \\ & & & X & X \\ \hline -2i & Z & \cdots & Z & X \end{array} \qquad ❌ \quad \begin{array}{cccc} & ? & \cdots & ? \\ Z & ? & & \\ \hline Z & \cdots & Z & X \end{array}$$

$\boldsymbol{A}^k = Z\cdots Z$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots YX$

$$\begin{array}{rcccc} & Z & \cdots & Z & \\ & & & X & X \\ \hline +2i & Z & \cdots & Y & X \end{array}$$

Proof: $A^k = Z\cdots$ case

$$H=\sum_{i=1}^{N}\left(J_X X_i X_{i+1}+J_Y Y_i Y_{i+1}+h_X X_i+h_Y Y_i+h_Z Z_i\right)\ \text{with}\ \begin{array}{c} J_X\neq 0, J_Y\neq 0 \\ (h_X,h_Y)\neq(0,0)\end{array}$$

By similar discussions,

$\boldsymbol{A}^k = Z\cdots Y$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZX$

$$\begin{array}{rcccc} & Z & \cdots & Y & \\ & & & X & X \\ \hline -2i & Z & \cdots & Z & X \end{array} \qquad \begin{array}{cccc} ? & \cdots & ? & \\ & & & \\ \color{red}{Z} & ? & & \\ \hline Z & \cdots & Z & X \end{array}$$

$\boldsymbol{A}^k = Z\cdots Z$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots YX$

$$\begin{array}{rcccc} & Z & \cdots & Z & \\ & & & X & X \\ \hline +2i & Z & \cdots & Y & X \end{array} \qquad \begin{array}{cccc} ? & \cdots & ? & \\ & & & \\ \color{red}{Z} & ? & & \\ \hline Z & \cdots & Y & X \end{array}$$

Proof: $A^k = Z\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

By similar discussions,

$\boldsymbol{A}^k = Z\cdots Y$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots ZX$

$$\begin{array}{ccccc} & Z & \cdots & Y & \\ & & & X & X \\ \hline -2i & Z & \cdots & Z & X \end{array} \qquad \times \quad \begin{array}{cccc} ? & \cdots & ? & \\ Z & ? & & \\ \hline Z & \cdots & Z & X \end{array}$$

$\boldsymbol{A}^k = Z\cdots Z$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = Z\cdots YX$

$$\begin{array}{ccccc} & Z & \cdots & Z & \\ & & & X & X \\ \hline +2i & Z & \cdots & Y & X \end{array} \qquad \times \quad \begin{array}{cccc} ? & \cdots & ? & \\ Z & ? & & \\ \hline Z & \cdots & Y & X \end{array}$$

$q_{\boldsymbol{A}_i^k} = 0$ if $\boldsymbol{A}^k = Z\cdots$

Proof: $A^k = XX\cdots, XI\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

By similar discussions,

$A^k = XX\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XX\cdots ZY$

	X	X	$\cdots$	X	
				Y	Y
$+2i$	X	X	$\cdots$	Z	Y

	?	$\cdots$	Z	Y
X	X			
X	X	$\cdots$	Z	Y

$A^k = XI\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XI\cdots ZY$

	X	I	$\cdots$	X	
				Y	Y
$+2i$	X	I	$\cdots$	Z	Y

	?	$\cdots$	Z	Y
X	X			
X	I	$\cdots$	Z	Y

Proof: $A^k = XX\cdots, XI\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

By similar discussions,

$\boldsymbol{A}^k = XX\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XX\cdots ZY$

$$\begin{array}{ccccc} & X & X & \cdots & X & \\ & & & & Y & Y \\ \hline +2i & X & X & \cdots & Z & Y \end{array} \qquad \begin{array}{ccccc} ✖ & ? & \cdots & Z & Y \\ {\color{red}X} & {\color{red}X} & & & \\ \hline X & X & \cdots & Z & Y \end{array}$$

$\boldsymbol{A}^k = XI\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XI\cdots ZY$

$$\begin{array}{cccccc} & X & I & \cdots & X & \\ & & & & Y & Y \\ \hline +2i & X & I & \cdots & Z & Y \end{array} \qquad \begin{array}{ccccc} ✖ & ? & \cdots & Z & Y \\ {\color{red}X} & {\color{red}X} & & & \\ \hline X & I & \cdots & Z & Y \end{array}$$

$q_{\boldsymbol{A}_i^k} = 0$ if $\boldsymbol{A}^k = XX\cdots, XI\cdots$

Proof: $A^k = XY\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

$\boldsymbol{A}^k = XY\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XY\cdots ZY$

$$\begin{array}{ccccc} & X & Y & \cdots & X & \\ & & & & Y & Y \\ \hline +2i & X & Y & \cdots & Z & Y \end{array} \qquad \begin{array}{cccccc} & & Z & \cdots & Z & Y \\ & X & X & & & \\ \hline +2i & X & Y & \cdots & Z & Y \end{array}$$

Proof: $A^k = XY\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

$\boldsymbol{A}^k = XY\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XY\cdots ZY$

$$\begin{array}{cccccc} & X & Y & \cdots & X & \\ & & & & Y & Y \\ \hline +2i & X & Y & \cdots & Z & Y \end{array} \qquad \begin{array}{cccccc} & & Z & \cdots & Z & Y \\ & X & X & & & \\ \hline +2i & X & Y & \cdots & Z & Y \end{array}$$

$$+2i\, q_{(XY\cdots X)_i} J_Y + 2i\, q_{(Z\cdots ZY)_{i+1}} J_X = 0$$

Proof: $A^k = XY\cdots$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \text{ with } \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

$\boldsymbol{A}^k = XY\cdots X$ has zero coeff.

$\because$ Consider $\boldsymbol{B}^{k+1} = XY\cdots ZY$

$$\begin{array}{cccccc} & X & Y & \cdots & X & \\ & & & & Y & Y \\ \hline +2i & X & Y & \cdots & Z & Y \end{array} \qquad \begin{array}{cccccc} & & Z & \cdots & Z & Y \\ & X & X & & & \\ \hline +2i & X & Y & \cdots & Z & Y \end{array}$$

$$+2i\, q_{(XY\cdots X)_i} J_Y + 2i\, q_{(Z\cdots ZY)_{i+1}} J_X = 0$$

Shown as zero on prev. slide

$$q_{\boldsymbol{A}_i^k} = 0 \text{ if } \boldsymbol{A}^k = XY\cdots$$

Current status of proof

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

$X \cdots$

$Y \cdots$

$Z \cdots$

zero coeff.

not proven yet

Current status of proof

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

$X\cdots$

$Y\cdots$

$Z\cdots$

zero coeff.

not proven yet

Current status of proof

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

zero coeff.

$XX\cdots$
$XI\cdots$
$XY\cdots$

$Z\cdots$

not proven yet

$XZ\cdots$

$Y\cdots$

Current status of proof

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

$XX\cdots$
$XI\cdots$
$XY\cdots$

$YY\cdots$
$YI\cdots$
$YX\cdots$

$Z\cdots$

zero coeff.

$XZ\cdots$

$YZ\cdots$

not proven yet

Current status of proof

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i\right) \text{ with } \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

zero coeff.

$XX\cdots$ $XZX\cdots$
$XI\cdots$ $XZI\cdots$
$XY\cdots$ $XZY\cdots$

$YY\cdots$ $YZY\cdots$
$YI\cdots$ $YZI\cdots$
$YX\cdots$ $YZX\cdots$

$Z\cdots$

not proven yet

$XZZ\cdots$

$YZZ\cdots$

Current status of proof

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

All the other

zero coeff.

$XZZ \cdots ZZX$

$XZZ \cdots ZZY$

$YZZ \cdots ZZX$

$YZZ \cdots ZZY$

not proven yet

Linear relations of coeffs.

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \text{ with } \begin{matrix} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{matrix}$$

Remaining coeffs. satisfy the following linear relations

$$q_{(XZZ\cdots ZZX)_i} = \frac{J_X}{J_Y} q_{(YZZ\cdots ZZY)_{i+1}} = q_{(XZZ\cdots ZZX)_{i+2}} = \cdots$$

$$q_{(XZZ\cdots ZZY)_i} = - q_{(YZZ\cdots ZZX)_{i+1}} = q_{(XZZ\cdots ZZY)_{i+2}} = \cdots$$

Linear relations of coeffs.

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

Remaining coeffs. satisfy the following linear relations

$$\begin{array}{cccccc} & X & Z & \cdots & Z & X & \\ & & & & & Y & Y \\ \hline +2i & X & Z & \cdots & Z & Z & Y \end{array} \qquad \begin{array}{cccccc} & & Y & Z & \cdots & Z & Y \\ & X & X & & & & \\ \hline -2i & X & Z & Z & \cdots & Z & Y \end{array}$$

$$q_{(XZZ\cdots ZZX)_i} = \frac{J_X}{J_Y} q_{(YZZ\cdots ZZY)_{i+1}} = q_{(XZZ\cdots ZZX)_{i+2}} = \cdots$$

$$q_{(XZZ\cdots ZZY)_i} = -q_{(YZZ\cdots ZZX)_{i+1}} = q_{(XZZ\cdots ZZY)_{i+2}} = \cdots$$

Linear relations of coeffs.

$$H=\sum_{i=1}^{N}\left(J_X X_i X_{i+1}+J_Y Y_i Y_{i+1}+h_X X_i+h_Y Y_i+h_Z Z_i\right)$$ with $$J_X\neq 0, J_Y\neq 0$$ $$(h_X,h_Y)\neq(0,0)$$

Remaining coeffs. satisfy the following linear relations

$$\begin{array}{ccccccc} & X & Z & \cdots & Z & X & \\ & & & & & Y & Y \\ \hline +2i & X & Z & \cdots & Z & Z & Y \end{array} \qquad \begin{array}{ccccccc} & & Y & Z & \cdots & Z & Y \\ & X & X & & & & \\ \hline -2i & X & Z & Z & \cdots & Z & Y \end{array}$$

$$q_{(XZZ\cdots ZZX)_i}=\frac{J_X}{J_Y}q_{(YZZ\cdots ZZY)_{i+1}}=q_{(XZZ\cdots ZZX)_{i+2}}=\cdots$$

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We have obtained all the equations of $\{q_{\boldsymbol{A}_i^k}\}$ corresponding to $\{r_{\boldsymbol{B}_i^{k+1}}=0\}$

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We have obtained all the equations of $\{q_{\boldsymbol{A}_i^k}\}$ corresponding to $\{r_{\boldsymbol{B}_i^{k+1}}=0\}$

Next: $\{r_{\boldsymbol{B}_i^k}=0\}$

Proof: $A^k = YZZ\cdots ZZY$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

ex.) $k = 5$, $\boldsymbol{A}^k = YZZZY$ (complicated)

$\boldsymbol{B}^k = YZZYY$

$$\begin{array}{cccccc} & Y & Z & Z & Z & Y \\ & & & & X & \\ \hline +2i & Y & Z & Z & Y & Y \end{array} \qquad \begin{array}{cccccc} & & X & Z & Y & Y \\ & Y & Y & & & \\ \hline +2i & Y & Z & Z & Y & Y \end{array}$$

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$$\boldsymbol{B}^k = XZYZX \qquad \begin{array}{cccccc} & X & Z & Z & Z & X \\ & & & X & & \\ \hline +2i & X & Z & Y & Z & X \end{array} \qquad \begin{array}{cccccc} & X & Z & Y & Y & \\ & & & & X & X \\ \hline -2i & X & Z & Y & Z & X \end{array} \qquad \begin{array}{cccccc} & & Y & Z & Z & X \\ & X & X & & & \\ \hline -2i & X & Z & Y & Z & X \end{array}$$

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$\boldsymbol{B}^k = YYZZY$

$$\begin{array}{cccccc} & Y & Z & Z & Z & Y \\ & & X & & & \\ \hline +2i & Y & Y & Z & Z & Y \end{array} \qquad \begin{array}{cccccc} & Y & Y & Z & X & \\ & & & & Y & Y \\ \hline +2i & Y & Y & Z & Z & Y \end{array}$$

Proof: $A^k = YZZ\cdots ZZY$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

ex.) $k = 5$, $\boldsymbol{A}^k = YZZZY$

$$+2i\, h_X\, q_{(YZZZY)_i} \qquad +2i\, J_Y\, q_{(XZYY)_{i+1}} = 0$$

$$+2i\, h_X\, q_{(XZZZX)_{i+1}} \quad -2i\, J_X\, q_{(XZYY)_{i+1}} \quad -2i\, J_Y\, q_{(YYZX)_{i+2}} = 0$$

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$$q_{(XZZZX)_{i+1}} = J_X / J_Y\, q_{(YZZZY)_i}$$

$$+2i\, h_X\, q_{(YZZZY)_{i+2}} \quad +2i\, h_X\, q_{(YYZX)_{i+2}} = 0$$

$$q_{(YZZZY)_{i+2}} = q_{(YZZZY)_i}$$

Proof: $A^k = YZZ\cdots ZZY$ case

$$H = \sum_{i=1}^{N} \left(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_X X_i + h_Y Y_i + h_Z Z_i \right) \quad \text{with} \quad \begin{array}{c} J_X \neq 0, J_Y \neq 0 \\ (h_X, h_Y) \neq (0,0) \end{array}$$

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$$= J_X / J_Y q_{(YZZZY)_i}$$

$$+) \quad +2i\, h_X\, q_{(YZZZY)_{i+2}} \qquad +2i\, h_X\, q_{(YYZX)_{i+2}} \qquad = 0$$

$$= q_{(YZZZY)_i}$$

$$\boldsymbol{+6i\, h_X\, q_{(YZZZY)_i} = 0}$$

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ex.) $k = 5$, $A^k = YZZZY$

$$+2i\, h_X\, q_{(YZZZY)_i} \qquad\qquad +2i\, J_Y\, q_{(XZYY)_{i+1}} = 0$$

$$+2i\, h_X\, q_{(XZZZX)_{i+1}} \quad -2i\, J_X\, q_{(XZYY)_{i+1}} \quad -2i\, J_Y\, q_{(YYZX)_{i+2}} = 0 \quad \times \frac{J_Y}{J_X}$$

$= J_X / J_Y q_{(YZZZY)_i}$

$$+)\quad +2i\, h_X\, q_{(YZZZY)_{i+2}} \quad +2i\, h_X\, q_{(YYZX)_{i+2}} = 0$$

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In this way, we get $q_{A_i^k} = 0$ for all remaining A^k if $(h_X, h_Y) \neq (0,0)$

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ex.) $k = 5$, $\boldsymbol{A}^k = YZZZY$

$+2i\, h_X\, q_{(YZZZY)_i}$ $\quad +2i\, J \ldots \quad 0$

$+ \ldots \quad \times \dfrac{J_Y}{J_X}$

k-local conserved quantity is absent!!

$+)\quad + \ldots (YZZZY)_{i+2} \quad +2i\, h_X\, q_{(YYZX)_{i+2}} \quad = 0$

$= q_{(YZZZY)_i}$

$$\boldsymbol{+6i\, h_X\, q_{(YZZZY)_i} = 0}$$

In this way, we get $q_{\boldsymbol{A}_i^k} = 0$ for all remaining $\boldsymbol{A}^k$ if $(h_X, h_Y) \neq (0,0)$

Summary: S=1/2

Main Theorem [Yamaguchi, Chiba, Shiraishi arXiv:2411.02162]
General spin-1/2 chains with symmetric n.n. interaction
do not have nontrivial local conserved quantities,
except for known integrable systems and their equivalents

Almost all systems in this class are non-integrable

0	$\sum_{i=1}^{N} h_Z Z_i$		I	(trivial)
1	$\sum_{i=1}^{N} \begin{pmatrix} J_Z Z_i Z_{i+1} \\ +h_X X_i \\ +h_Z Z_i \end{pmatrix}$	$h_X = 0$	I	(trivial)
		$h_X \neq 0, h_Z = 0$	I	transverse field Ising
		$h_X \neq 0, h_Z \neq 0$	N	[Chiba]
2	$\sum_{i=1}^{N} \begin{pmatrix} J_X X_i X_{i+1} \\ +J_Y Y_i Y_{i+1} \\ +h_X X_i \\ +h_Z Z_i \\ +h_Z Z_i \end{pmatrix}$	$(h_X, h_Y) = (0,0)$	I	XY
		$(h_X, h_Y) \neq (0,0)$	N	our result

3	$\sum_{i=1}^{N} \begin{pmatrix} J_X X_i X_{i+1} \\ +J_Y Y_i Y_{i+1} \\ +J_Z Z_i Z_{i+1} \\ +h_X X_i \\ +h_Z Z_i \\ +h_Z Z_i \end{pmatrix}$	$J_X = J_Y = J_Z$		I	XXX (Heisenberg)
		$J_X = J_Y \neq J_Z$ (two are equal)	$(h_X, h_Y) = (0,0)$	I	XXZ
			$(h_X, h_Y) \neq (0,0)$	N	our result
		J_X, J_Y, J_Z all different	$(h_X, h_Y, h_Z) = \vec{0}$	I	XYZ
			$(h_X, h_Y, h_Z) \neq \vec{0}$	N	our result

Contents

Model & Result

Spin-1 bilinear-biquadratic chain

$$H = \sum_{i=1}^{N} J_1 \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right) + J_2 \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^2$$

Including **AKLT** $(J_2 = J_1/3)$ and **Heisenberg** $(J_2 = 0)$

Integrable: $J_1 = 0, \pm J_2$

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Main Theorem [Park, Lee 2410.23286] [Hokkyo, Yamaguchi, Chiba 2411.04945]

It is **non-integrable**, except for known integrable systems

(do not have nontrivial local conserved quantities)

Model & Result

Spin-1 bilinear-biquadratic chain

$$H = \sum_{i=1}^{N} J_1 \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right) + J_2 \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^2 + D\left(S_i^z\right)^2$$

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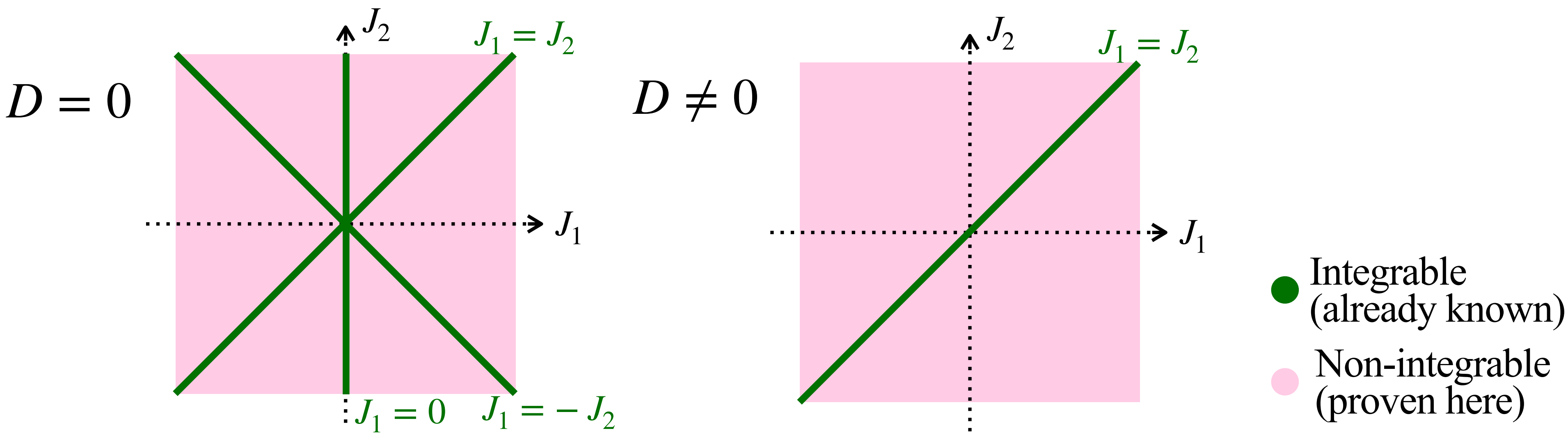

Basis

In non-integrability proof, basis choice is important!

Basis

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The extension to spin-1 systems such as the AKLT model [55] looks not straightforward since the rule of the product of spin-1 operators is more complicated than the case of spin-1/2.

[Shiraishi (2019)]

Basis

In non-integrability proof, basis choice is important!

Basis (for 3×3 matrix) is desired to

- have simple commutation relations
- describe the model Hamiltonian simply

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Basis (for 3×3 matrix) is desired to

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- describe the model Hamiltonian simply

We have found a basis that **satisfies both**

$$E^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E^{+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^{-1} = \left(E^{+1}\right)^\dagger,$$

$$F^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^{+2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F^{-1} = \left(F_{+1}\right)^\dagger, \quad F^{-2} = \left(F^{+2}\right)^\dagger.$$

Basis

Commutator table

a \ b	E_{+1}	E_0	E_{-1}	F_{+2}	F_{+1}	F_0	F_{-1}	F_{-2}
E_{+1}	0	$-E_{+1}$	$+E_0$	0	$-2F_{+2}$	$-3F_{+1}$	$+F_0$	$+F_{-1}$
E_0	$+E_{+1}$	0	$-E_{-1}$	$+2F_{+2}$	$+F_{+1}$	0	$-F_{-1}$	$-2F_{-2}$
E_{-1}	$-E_0$	$+E_{-1}$	0	$-F_{+1}$	$-F_0$	$+3F_{-1}$	$+2F_{-2}$	0
F_{+2}	0	$-2F_{+2}$	$+F_{+1}$	0	0	0	$-E_{+1}$	$+E_0$
F_{+1}	$+2F_{+2}$	$-F_{+1}$	$+F_0$	0	0	$-3E_{+1}$	$+E_0$	$-E_{-1}$
F_0	$+3F_{+1}$	0	$-3F_{-1}$	0	$+3E_{+1}$	0	$-3E_{-1}$	0
F_{-1}	$-F_0$	$+F_{-1}$	$-2F_{-2}$	$+E_{+1}$	$-E_0$	$+3E_{-1}$	0	0
F_{-2}	$-F_{-1}$	$+2F_{-2}$	0	$-E_0$	$+E_{-1}$	0	0	0

Basis

Commutator table

b \ a	E_{+1}	E_0	E_{-1}	F_{+2}	F_{+1}	F_0	F_{-1}	F_{-2}
E_{+1}	0	$-E_{+1}$	$+E_0$	0	$-2F_{+2}$	$-3F_{+1}$	$+F_0$	$+F_{-1}$
E_0	$+E_{+1}$	0	$-E_{-1}$	$+2F_{+2}$	$+F_{+1}$	0	$-F_{-1}$	$-2F_{-2}$
E_{-1}	$-E_0$	$+E_{-1}$	0	$-F_{+1}$	$-F_0$	$+3F_{-1}$	$+2F_{-2}$	0
F_{+2}	0	$-2F_{+2}$	$+F_{+1}$	0	0	0	$-E_{+1}$	$+E_0$
F_{+1}	$+2F_{+2}$	$-F_{+1}$	$+F_0$	0	0	$-3E_{+1}$	$+E_0$	$-E_{-1}$
F_0	$+3F_{+1}$	0	$-3F_{-1}$	0	$+3E_{+1}$	0	$-3E_{-1}$	0
F_{-1}	$-F_0$	$+F_{-1}$	$-2F_{-2}$	$+E_{+1}$	$-E_0$	$+3E_{-1}$	0	0
F_{-2}	$-F_{-1}$	$+2F_{-2}$	0	$-E_0$	$+E_{-1}$	0	0	0

Only single terms appear

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a \ b	E_{+1}	E_0	E_{-1}	F_{+2}	F_{+1}	F_0	F_{-1}	F_{-2}
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E_{-1}	$-E_0$	$+E_{-1}$	0	$-F_{+1}$	$-F_0$	$+3F_{-1}$	$+2F_{-2}$	0
F_{+2}	0	$-2F_{+2}$	$+F_{+1}$	0	0	0	$-E_{+1}$	$+E_0$
F_{+1}	$+2F_{+2}$	$-F_{+1}$	$+F_0$	0	0	$-3E_{+1}$	$+E_0$	$-E_{-1}$
F_0	$+3F_{+1}$	0	$-3F_{-1}$	0	$+3E_{+1}$	0	$-3E_{-1}$	0
F_{-1}	$-F_0$	$+F_{-1}$	$-2F_{-2}$	$+E_{+1}$	$-E_0$	$+3E_{-1}$	0	0
F_{-2}	$-F_{-1}$	$+2F_{-2}$	0	$-E_0$	$+E_{-1}$	0	0	0

Only single terms appear

$$H = \sum_i (J_1 - J_2/2)\left(E_i^0 E_{i+1}^0 + E_i^{+1} E_{i+1}^{-1} + E_i^{-1} E_{i+1}^{+1}\right)$$
$$+J_2/6\left(F_i^0 F_{i+1}^0 + 3F_i^{+1} F_{i+1}^{-1} + 3F_i^{-1} F_{i+1}^{+1} + 6F_i^{+2} F_{i+1}^{-2} + 6F_i^{-2} F_{i+1}^{+2}\right) + D/3\, F_i^0$$

The bilinear-biquadratic Hamiltonian is described simply

Contents

Other recent progress

Non-integrability proof is extended just now

- higher dimensional systems (general $d \geq 2$)

  d-dim. transverse-field Ising [Chiba (coming soon)]
  d-dim. XY / Heisenberg [Shiraishi, Tasaki (coming soon)]

- higher spin systems (general $S \geq 3/2$)

  [Hokkyo, Yamaguchi, Chiba (in prep.)]

- bosonic systems

  [Yamaguchi (in prep.)]

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The **presence or absence of 3-local** conserved quantities is coincident with **that of k-local** conserved quantities for **general** k

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All studies to date support this Conj.

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Proving $\{q_{\mathbf{A}_i^k} = 0\}$ for general k requires craftsmanship

Conjecture Related:[Grabowski, Mathieu] [Reshetikhin's condition]

The **presence or absence of 3-local** conserved quantities is coincident with **that of k-local** conserved quantities for **general** k

≒ Partially integrable system does not exist

(with finite number of local conserved quantities)

All studies to date support this Conj.

Admitting this Conj., integrability test can be performed with a low-cost algorithm (on Ker of $[\bullet, H]$ in finite-dim. linear space)

Summary

Classification of integrability and non-integrability is given for general spin-1/2 chains with symmetric n.n. interaction and spin-1 bilinear-biquadratic chains

We prove that all systems are non-integrable,
except for known integrable systems,
which implies the ubiquitousness of non-integrability

Trivial local conserved quantities

Usually, non-integrable systems have H
as the only local conserved quantity

However, some non-integrable systems
have other k-local conserved quantities with $k = 1,2$

The following is a complete list

$$H = \sum_{i=1}^{N} \left(J_X \left(X_i X_{i+1} - \left(\frac{h_Y}{h_X} \right)^2 Y_i Y_{i+1} \right) + h_X X_i + h_Y Y_i \right) ,$$

$$Q = \sum_{i=1}^{N} (-1)^i (h_X X_i + h_Y Y_i)(h_X X_{i+1} + h_Y Y_{i+1}) \qquad (54)$$

$$H = \sum_{i=1}^{N} (J_X (X_i X_{i+1} - Y_i Y_{i+1}) + J_Z Z_i Z_{i+1} + h_Z Z_i) ,$$

$$Q = \sum_{i=1}^{N} (-1)^i Z_i , \qquad (55)$$

result on spin-1/2 systems

Theorem 2. *In the BLBQ model (1) not satisfying Eq. (2), k-local conserved quantities with $k \leq N/2$ are restricted to linear combinations of the following:*

(i) its own Hamiltonian: H,

(ii) the total magnetization in the z direction: $M_z = \sum_{i=1}^{N} S_i^z$,

(iii) the total magnetizations in the x and y directions: M_x and M_y, if $D = 0$ holds,

(iv) the staggered quadratic spins:

$$\sum_{i=1}^{N} (-1)^i (S_i^z)^2 ,$$
$$\sum_{i=1}^{N} (-1)^i ((S_i^x)^2 - (S_i^y)^2) ,$$
$$\sum_{i=1}^{N} (-1)^i (S_i^x S_i^y + S_i^y S_i^x) , \qquad (3)$$

if $J_1 = 0$ holds and N is even.

result on spin-1 systems